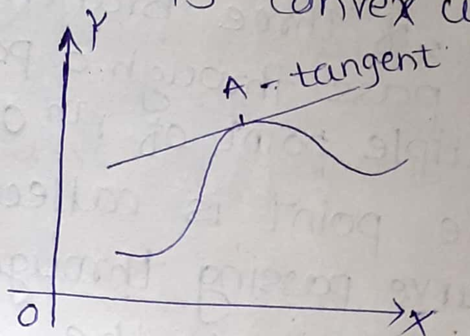
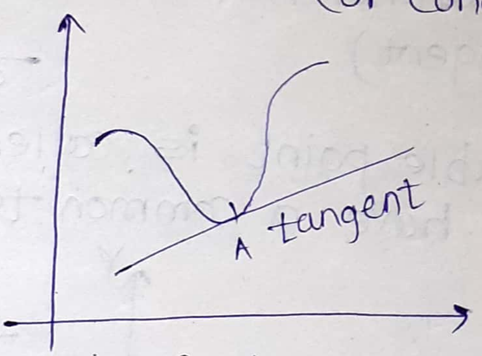
Curve Tracing & Rectification of curves.

Basic Defn :-

(1) convex upward:- If the portion of the curve on both sides of 'A' lies below the tangent at A then the curve is convex upward. (or concave downward).

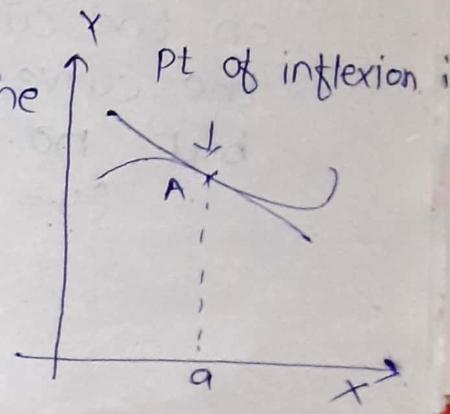


(2) convex Downward:- If the portion of the curve on both sides of 'A' lies above the tangent at A, then the curve is convex downward.
(or concave upward)



(3) singular point:- An unusual point on a curve is called a singular point such as, a point of inflexion, a double point, a middle point, cusp, node, or conjugate pt.

(4) point of inflexion:-
The point that separates the convex part of a continuous curve from the concave part is called the pt of inflexion of curve.
In other words, a pt where the curve unusually crosses its tangent is called a pt of inflexion.



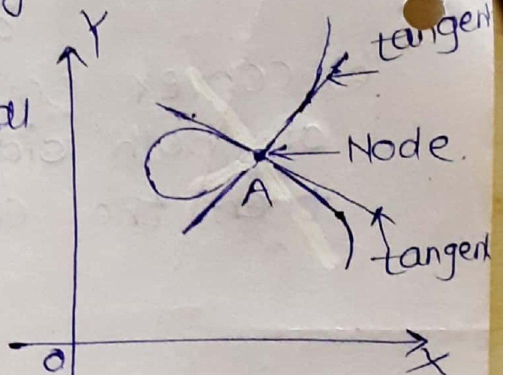
(5) Multiple point:- A point through which more tha one branches of a curve pass is called a Multipe point of the curve.

(6) A double point:- A point on a curve is called a double point, if two branches of the curve pass through it.

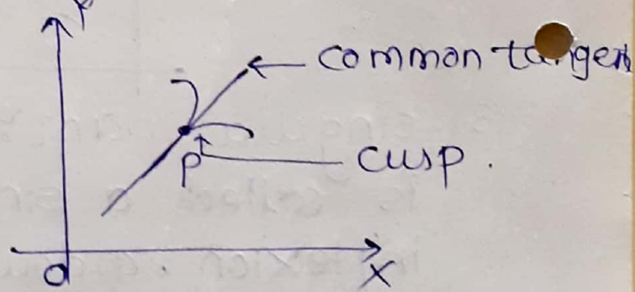
- A triple point, if three branches pass through it
- If r branches pass through a point, the point is called multiple point of r^{th} order.

(7) Node:- A double point is called node, if the branches of curve passing through it are real & tangents at the common pt. of intersection are real & distinct. (not coincident)

(if distinct branches have distinct tangent)

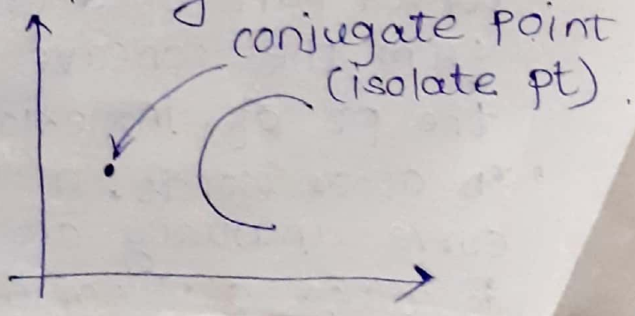


(8) Cusp:- A double point is called cusp if two branches have a common tangent.



(9) A conjugate point:- P is called a conjugate pt. on the curve if there are no real points on the curve, it satisfies the eqn of curve, but no branches pass through P.

($y = f(x)$)



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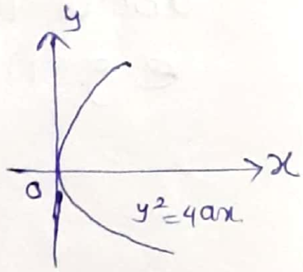
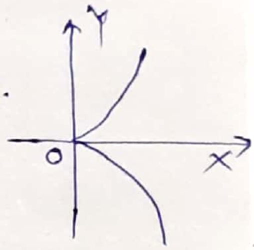
Type (I) Curves Given by cartesian equations. (Explicit Relations)

Rules for tracing of cartesian curves.

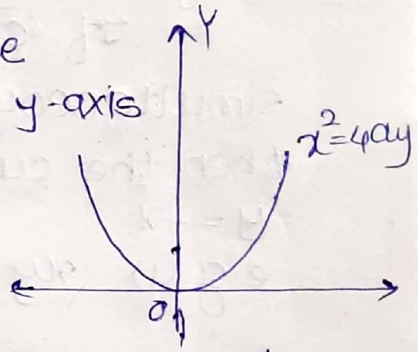
Rule (I) :- symmetry :-

(a) about x-axis :- on changing y by $-y$ if the equation remains unchanged. i.e. all the terms of y are even degree then the curve is symmetric about x-axis.
e.g. $y^2 = 4ax$, $y^2 = x^3$

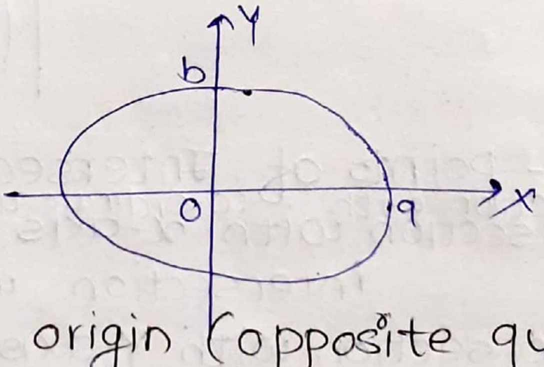




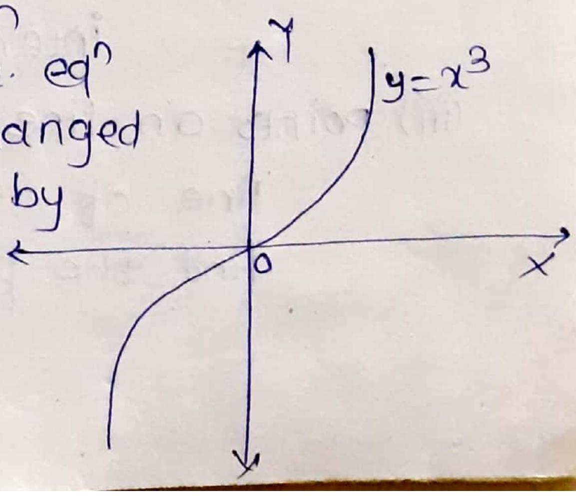
(b) about y-axis :- on changing x by $-x$ if the eq^n remains unchanged i.e. all the terms of x are even degree then the curve is symmetric about y-axis
e.g. $x^2 = 4ay$.



(c) about X & Y-axis :- If all the terms x and y are both even degree, then symmetry about both axis.
e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



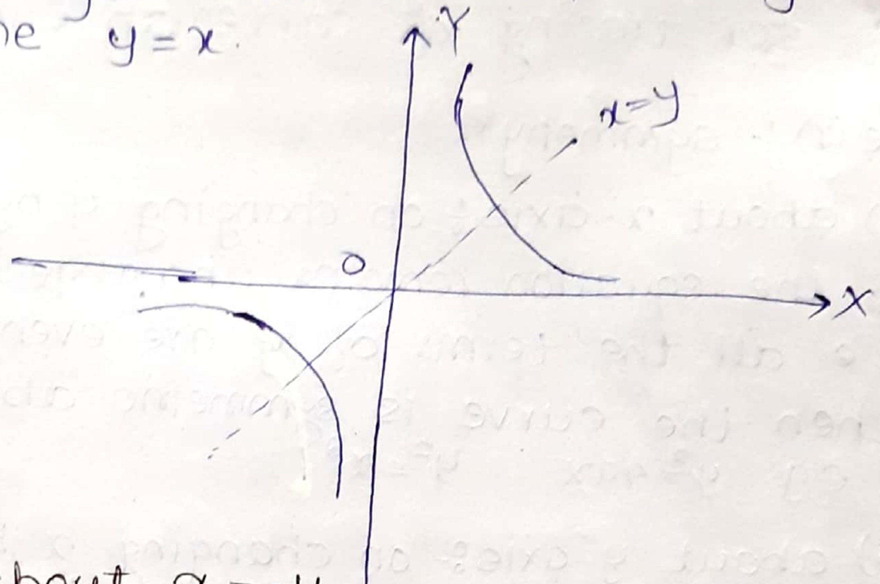
(d) symmetry about origin (opposite quadrants)
A curve is symmetrical in opposite quadrants, if the eq^n of the curve remains unchanged if x and y are replaced by $-x$ and $-y$ resp.
e.g. $y = x^3$



(e) symmetry about $x=y$

If on interchanging x and y, the equation remains unchanged then the curve is symmetric about the line $y=x$.

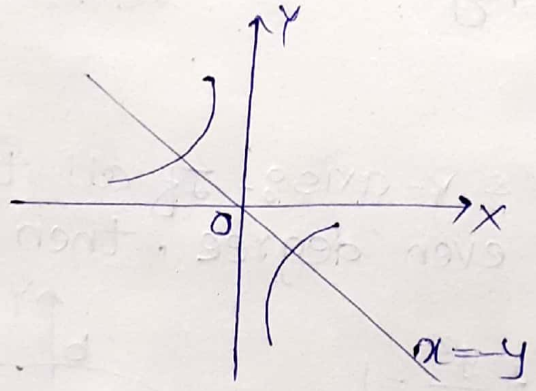
e.g. (i) $xy=c$



(f) symmetry about $x=-y$

If on replacing x by $-y$ and y by $-x$ simultaneously, the equation remains unchanged then the curve is symmetrical about the line $y=-x$

e.g. (i). $xy=-c$.



Rule 2:- Points of Intersection:-

(a) Intersection with co-ordinate axis:-

(i) Intersection with x-axis: put $y=0$ to find intersection with x-axis.

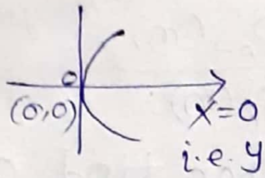
(ii) Intersection with Y-axis:- put $x=0$ to find intersection with y-axis.

(iii) points on line of symmetry:- If $y=x$ is the line of symmetry then put $y=x$ to find the points on line of symmetry.

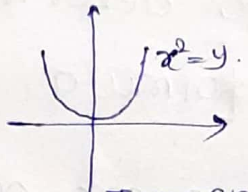
origins:- • when eq^n satisfies at $(0,0)$.
- It does not contain any absolute const.

(c) Tangents at the origin:- (To investigate nature of a multiple point)

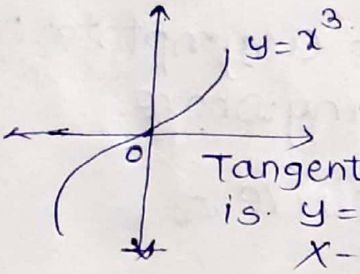
{ If $(0,0)$ is the pt on curve then.



i.e. y-axis.
Tangent at $(0,0)$ is $x=0$



Tangent at $(0,0)$ is $y=0$ i.e x axis.



Tangent at $(0,0)$ is $y=0$ i.e x-axis.

<u>Newton's Method</u>:- If a curve is given by a rational integral algebraic eq^n & passes through the origin, the eq^n of the tgt or tgts at the origin can be obtained by equating to zero the lowest degree terms taken together in the eq^n of curve.

Rule 38- Asymptotes:-
Asymptotes are tangents to the curve at infinity.

(i) Asymptotes parallel to x-axis:- equating to zero the coefficient of highest degree term in x.
e.g. $xy=1$ (if highest degree term is x is x only coeff-$y \Rightarrow y=0$) ||el to x-axis

(ii) Asymptotes parallel to y axis:- equating to zero the coefficient of highest degree term in y

(iii) <u>oblique Asymptotes</u>:-
Asymptotes which are not parallel to co-ordinate axes are called as oblique asymptotes

Method (1):-
Let $y=mx+c$ be the asymptote. The pt of intersection with the curve $f(x,y)=0$ are given by $f(x, mx+c)=0$. Equate to zero the coeff. of two successive highest power of x, giving eq^{ns} to determine m & c.

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Method 2:-

(i) Let $y = mx + c$ be the eqn of the asymptote.

(ii) Find $\phi_n(m)$ by putting $x = 1$ & $y = m$ in the highest degree (n) terms of the eqn.

(iii) Lly find $\phi_{n-1}(m)$.

(iv) solve $\phi_n(m) = 0$ to determine m.

(v) Find 'c' by the formula $c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)}$.

(vi) If the roots of m are equal, then find c by
$\frac{c^2}{2}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$.

(vii) If the Find more finite pts where asymptotes meets any other branch of the curve anywhere.

Rule (4):- Special points on the curve:-

(a) Find out such pts on the curve whose presence can be readily detected

(b) Find $\left(\frac{dy}{dx}\right)$ and the pts where the tangent is parallel to either axes.

(c) If $\left(\frac{dy}{dx}\right)_{P(x_1, y_1)} = 0$ then the tangent at $P(x_1, y_1)$ is parallel to x-axis.

d) If $\left(\frac{dy}{dx}\right)_{P(x_1, y_1)} = \infty$ then the tangent at $P(x_1, y_1)$ is parallel to Y-axis.

(e) Also find pt. of inflexion, double or multiple points, node, cusp, conjugate pt.

Rule (5):- Region of absence of the curve

(a) If possible, express the equation in the explicit form say $y = f(x)$ & examine how y varies as x varies continuously.

For $y = f(x)$, if y becomes imaginary for some value of $x > a$ (say), then no part of curve exists beyond $x = a$

- For $x = f(y)$, if x becomes imaginary for some value of $y > b$ (say) then no part of curve exists beyond $y = b$.

Some useful Remarks:-

(a) when we have for $y = f(x)$, put $x = 0$ & see what is y. Also observe how y varies as x increases from 0 to $+\infty$,

(b) If $y \to \infty$ as $x \to a$ then $x = a$ must be asymptote parallel to y-axis.

If $x \to \infty$ as $y \to b$ then $y = b$ must be asymptote parallel to x axis.

EX (1) Trace the curve $xy^2 = a^2(a-x)$

The given curve can be written as $y^2 = \dfrac{a^2(a-x)}{x}$

(1) symmetry = even powers of $y \Rightarrow$ symmetry about x-axis.

(2) ~~origin~~ pt of intersection:-

put $x = 0$ & $y = 0$ to find intersections with y & x-axis. Here $(a, 0)$ is the pt on the curve.

(3) Tangents at origin:- Curve does not pass through origin but cuts x-axis at $(a, 0)$. If we transfer the origin to the point $(a, 0)$, the new eq^n becomes $y^2(x+a) + a^2x = 0$

$\therefore$ origin is not the point on the curve.

(4) Tangent at any other point

To find the nature of the tangent at any point, find $\frac{dy}{dx}$ at that pt.

$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$. Thus the tangent at $(a,0)$ is ||el to y-axis.

(4) Asymptote:- $y^2 = \frac{a^2(a-x)}{x}$

As $x \to 0$, $y \to \infty$, hence the only asymptote is the line $x=0$ i.e. y-axis.

(5) Region of absence.

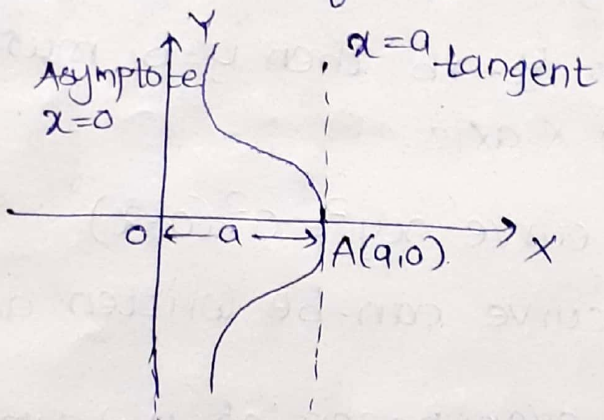
$$y = a\sqrt{\frac{a-x}{x}}$$

(i) At $x = -a$, $y^2 = -ve$

$x = 2a$, $y^2 = -ve$.

For $x<0$ & $x>a$, y becomes imaginary therefore the curve does not exist for $x<0$ & $x>a$.

Thus curve exists for $0<x<a$.



curves Given by parametric Equations:-

$$x = f(t), \quad y = g(t).$$

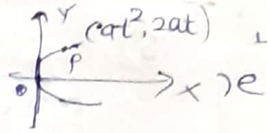
Rules:- (If possible, change to cartesian & then trace).

(1) Limitations of the curve:- Let the parametric eqⁿ be given by $x=f(t)$, $y=g(t)$. If possible find the greatest values of x & y for a proper value of t. & therefore the boundary lines parallel to x & y axes betⁿ which the curve lies.

(2) symmetry:-

(a) If $f(t)$ be even function of t & $g(t)$ an odd the curve is symmetrical to x-axis.

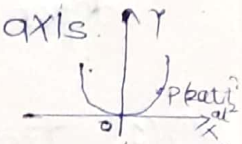
(i.e $f(-t) = f(t)$ & $g(t) = -g(t) \Rightarrow$ symmetry about x-axis) e.g. As the parabola $x = at^2$, $y = 2at \Rightarrow$ symmetry about x-axis.



(b) If $f(t)$ be odd and $g(t)$ an even function, symmetry about y-axis.

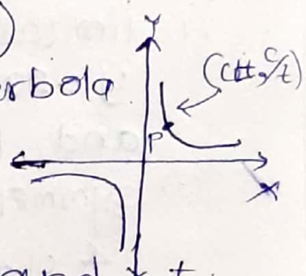
(i.e. $f(-t) = -f(t)$ & $g(-t) = g(t) \Rightarrow$ symmetry about y-axis)

e.g. Parabola $x = 2at$, $y = at$ has symmetry about y-axis.



(c) If both $f(t)$ & $g(t)$ are odd, the curve is symmetric in opposite quadrant then symmetry about opposite quadrant (i.e $f(-t) = -f(t)$ & $g(-t) = -g(t)$)

e.g. $x = ct$, $y = c/t$ the rectangular hyperbola



(d) Also we note that for value of t and $-t$ x remains unchanged but y has equal and opposite values, therefore the curve is symmetrical about x-axis e.g. $x = at^2$, $y = 2at$.

(e) Also we note that for values of t and $\pi - t$, y remains unchanged but x has equal and opposite values, hence the curve is symmetrical about y-axis.

e.g. $x = a\cos^3 t$, $y = a\sin^3 t$.

(3) origins:- If on putting $x = 0$ we obtain $y = 0$ for some value of t, then the curve & the axes. find asymptote if any.

(4) special points:- Try to find few points on the curve by observation and also those points where $\frac{dy}{dx} = 0$ or ∞. Here $x = f(t)$, $y = g(t)$. Hence use the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

(5) Region of absence of curve:- curve does not exist

(a) find those regions where

(b) make a table of values of $t, x, y, \frac{dx}{dt}$ & $\frac{dy}{dt}$.

(c) If both $x + y$ are periodic function of t, with a common period, we need to study the position of curve for one period only.

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(6) cycloid :- when a circle rolls in a plane along given straight line, the locus traced out by a fixed point on the circumference of rolling circle is called as cycloid.

The sketching of the cycloid from its equation depends on the values of x, y and $\frac{dy}{dx}$ at $t=0$.

Q(1) Trace the cycloid, $x = a(t+\sin t)$, $y=a(1-\cos t)$.

Sol^n:- The curve is known as cycloid.

(1) limitations:- The curve lies betn the lines $y=0$ & $y=2a$, because the greatest value of y is 2a and least is 0.

(2) symmetry:- $x=a(t+\sin t)$, being odd function of t. & $y=a(1-\cos t)$, an even function, the curve is symmetrical to y-axis.

(3) origin:- when $t=0$, $x=0$, and $y=0$, hence curve passes through the origin. The curve cuts X-axis. (Put $y=0$) when $y=a(1-\cos t)=0$ or $t=0$ then $x=0$

i.e. at (0, 0) only., similarly, it cuts Y-axis also at (0,0).

(4) No Asymptotes.

(5) special points :- we have after differentiat^n.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \tan\frac{t}{2}.$$

$\Rightarrow \frac{dy}{dx} = 0$ when $\frac{t}{2} = 0$ or when $t=0$.

& $\frac{dy}{dx} = \infty$ at $\tan\frac{t}{2} = \infty$ or at $t = \pi$.

Hence at (0,0), tangent is parallel to x-axis.

& at $(a\pi, 2a)$ tangent is parallel to Y-axis.

(6) Region of absence (a):-

Also $y = 2a\sin^2 t/2$ or $\sin t/2 = \sqrt{y/2a}$.

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hence when y is $-ve$, t is imaginary, i.e no part of the curve lies below x-axis (i.e. in 3rd & 4th quadrants)

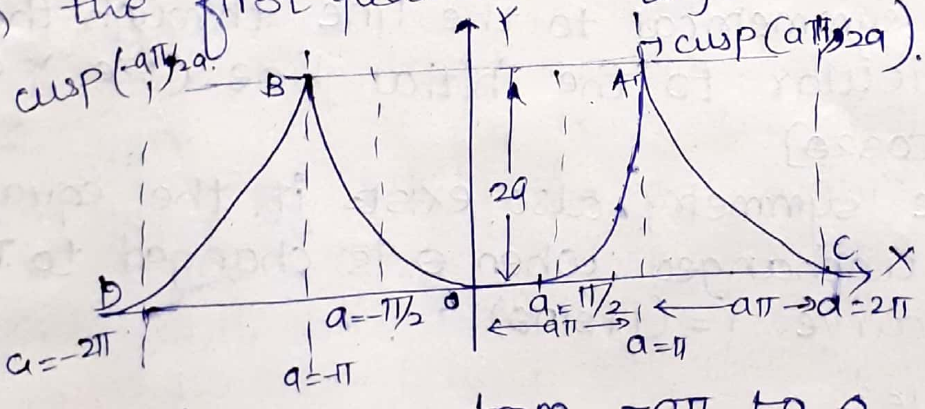
(7) The table of values of $t, x, y, \frac{dy}{dx}$ is as follows.

t	0	$\pi/2$	π	2π	$-\pi/2$	$-\pi$
x	0	$a(\pi/2+1)$	$a\pi$	$2a\pi$	$-a(\pi/2+1)$	$-a\pi$
y	0	a	$2a$	0	a	$2a$
dy/dx	0	1	∞	0	-1	$-a$

Hence some special points that lie on the curve ar $(0,0)$, $[a(\pi/2+1), a]$, $(a\pi, 2a)$, $(2a\pi, 0)$, $[-a(\pi/2+1), a]$ & $(-a\pi, 2a)$

$\Rightarrow$ x increases from $0 \to a\pi$. y also increases from $0 \to 2a$. But as x further increases to $2a\pi$, y decreases again to zero.

When t is $+ve$ & varies from $0 \to 2\pi$, we can sketch the portion OAC of the curve in the first quadrant.



As x increases from $-a\pi$ to 0, y is found to decrease from $2a \to 0$. Also the portion DBO can be trace by symmetry.

The curve consist of congruent arches extending to infinity in both direction of x-axis.

The curve is as traced below.

- The points A, B are called the cusps of the cycloid.
- The line OY (i.e. Y-axis) about which the curve is symmetrical is called the axis of the cycloid.
- The line AB joining the cusps is called the base

of cycloid.

- The point O is called the vertex of the cycloid.

← Pole ← Initial line

O(origin) → x

Polar co-ordinates (r, θ), $r = f(\theta)$

curves Given by polar co-ordinates:-

Rule (1):- symmetry:- In polar co-ordinates, the curve is often given by the eqn $r = f(\theta)$

(a) If the equation to the curve remains unchanged by changing θ to $-\theta$, it will be symmetrical to the initial line.
[e.g. $r = a(1+\cos\theta)$]

(b) If the equation of the curve remains unchanged by changing r to $-r$, the curve is symmetrical to the pole. In such a case only even power of r will occur in eqn [$r^2 = a^2\cos 2\theta$]

(c) If the equation remains unchanged by changing θ to $-\theta$ and r to $-r$, at the same time, the curve is symmetrical to the line through the pole, perpendicular to the initial line (i.e. y axis)
[$r^2 = a^2\cos 2\theta$]

The same symmetry also exists if the equation remains unchanged when θ is changed to $\pi - \theta$.
e.g the curve $r = (1+\sin\theta)$

Rule (2):- pole:-

(a) The pole will lie on the curve if for some value of θ, r becomes zero.

(b) Then find the equation of tangents at the pole. If we put $r = 0$, the value of θ gives the tangent at the pole.

care should be taken of the points where the curve cuts the initial line $\theta = \pi/2$

Rule (3):- The table showing value of r for diffn values of θ is very useful in plotting a polar curve. Also find the values of θ at which $r = 0$, or $r = \infty$.

Rule 4:- Angle between the radius vector and tangent [ϕ]

use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ and also the points where $\phi = 0$ or ∞.

i.e. find the points where the tangent coincides with the radius vector or is perpendicular to it.

Rule 5:- Asymptotes: Find Asymptote if any.

Rule (6):- Region of absence of the curve:-

(1) solve the eq^n for r & consider how r varies as θ increases from 0 to $+\infty$. and also when θ decreases from 0 to $-\infty$. If necessary from a table of a values of r and θ.

(2) If for some values of θ, say α and β, r^2 is negative, i.e. r imaginary, this means that no branch of the curve exists between line $\theta=\alpha$ & $\theta=\beta$.

(3) If the maximum numerical value of r is a, the entire curve will ~~be~~ lie within a circle of radius a (i.e. $r=a$). If least numerical value of r ~~and~~ is b. the curve will lie outside the circle $r=b$.

(4) In most of the polar eq^n, only periodic funct^{ns} $\sin\theta$ & $\cos\theta$ occur & hence value θ from $0 \to 2\pi$ should only be considered. The remaining values of θ, give no new branch of the curve.

Ex:- Trace the curve $r^2 = a^2\cos 2\theta$.

Sol^n:- (1) ~~The curve is sy~~ If we change $\theta \to -\theta$ eq^n remains unchanged

$\therefore$ curve is symmetrical to the initial line.

(2) If we change $r \to -r$ eq^n remains unchanged

$\therefore$ curve is symmetrical to the pole.

from ① + ② we can say curve is symmetrical to the perpendicular line to the initial line passing through origin i.e pole

(3) when $\theta = \pi/4$ & $\theta = 3\pi/4$, r becomes zero.

∴ curve pass through the pole

(put $r=0 \Rightarrow \cos 2\theta = 0$. (tangents at pole

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$) ← are obtained by putting $r=0$)

(4). variation of r corresponding to θ.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$7\pi/4$	2π
r	a	0	Imgi	0	-a	0	0	a

∴ r is maximum = a.

when $\theta = 0$, $r = \pm a$ i.e. curve pass through pts $(a,0)$, & $(-a,0)$. minimum value of r is 0.

(5) As θ increases $\pi/4 \to 3\pi/4$, r^2 remains negative for $\pi/4 < \theta < 3\pi/4$ & the curve does not exist in this region. IIly curve does not exist in $5\pi/4 < \theta < 7\pi/4$.

(6) Angle ϕ :-

we use the formula

$$\tan\phi = r\frac{d\theta}{dr}$$

$$r^2 = a^2\cos 2\theta.$$

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta.$$

$$\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$$

$$\frac{d\theta}{dr} = \frac{-r}{a^2\sin 2\theta}$$

$$r\frac{d\theta}{dr} = \frac{-r^2}{a^2\sin 2\theta}.$$

$$= \frac{-a^2\cos 2\theta}{a^2\sin 2\theta} = -\cot 2\theta.$$

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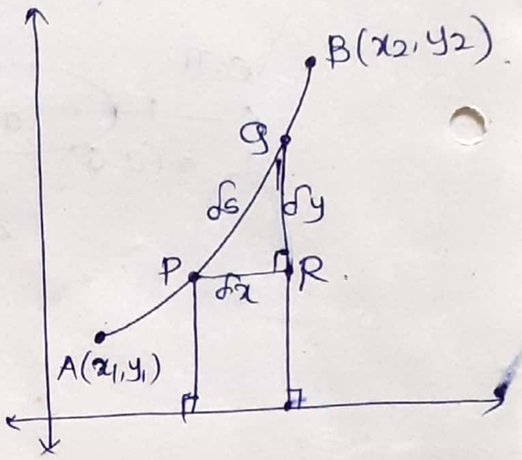
Rectification of curves.

The process of determination of the legths of the arcs of the plane curves whose equations are given in cartesian, parametric and polar form is known as Rectification.

Rectification of plane curve for cartesian eqn.

(1) $y = f(x)$

consider two points $P(x, y)$ and $Q = (x+\delta x, y+\delta y)$ on the curve, arc $PQ = \delta s$.



from right angled triangle PQR.

$$(PQ)^2 = (PR)^2 + (RQ)^2$$

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

dividing by $(\delta x)^2$

we get $\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$

as limit $\delta x \to 0$ i.e $P \to Q$ we have.

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{---(1)}$$

Integrating (1) w.r.t x from $A(x_1, y_1)$ to $B(x_2, y_2)$ we get the formula for the length of the curve from A to B as.

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

(2) $x = f(y)$

similarly if $x = f(y)$ then.

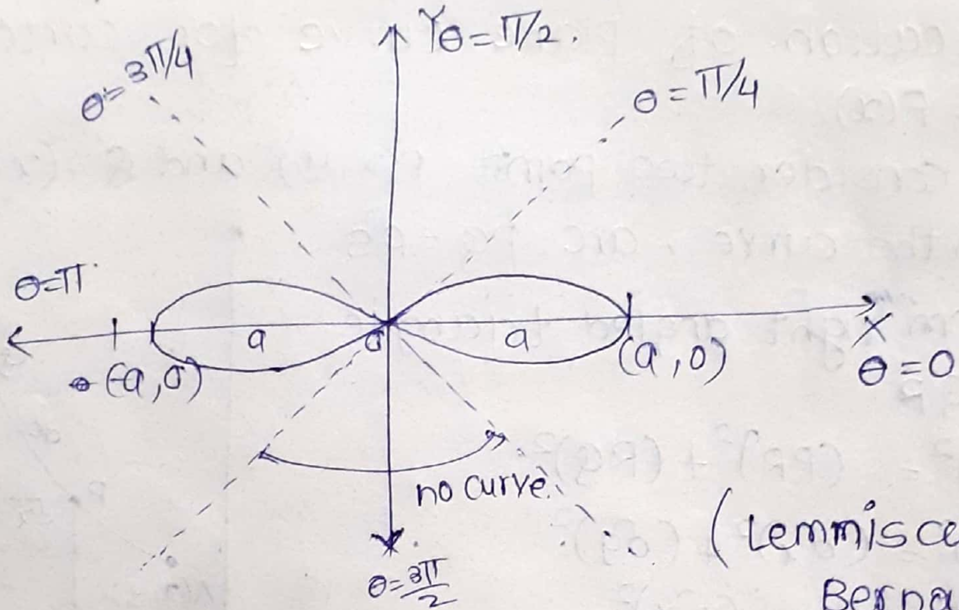
$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

$\therefore \tan\phi = -\cot 2\theta.$

$\tan\phi = -\tan(\pi/2 + 2\theta)$

$\therefore \phi = \frac{\pi}{2} + 2\theta.$

Hence at $\theta = 0$, $\theta = \pi/2$ i.e. tangent is $\perp^{ar}$ to the initial line at the points $(a, 0)$ & $(-a, 0)$



$\therefore$ (Lemniscate of Bernoulli).